

ON THE REPRESENTATIONS OF A POSITIVE INTEGER BY CERTAIN CLASSES OF QUADRATIC FORMS IN EIGHT VARIABLES

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ABSTRACT. In this paper we use the theory of modular forms to find formulas for the number of representations of a positive integer by certain class of quadratic forms in eight variables, viz., forms of the form $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 + b_1(x_5^2 + x_5x_6 + x_6^2) + b_2(x_7^2 + x_7x_8 + x_8^2)$, where $a_1 \leq a_2 \leq a_3 \leq a_4$, $b_1 \leq b_2$ and a_i 's $\in \{1, 2, 3\}$, b_i 's $\in \{1, 2, 4\}$. We also determine formulas for the number of representations of a positive integer by the quadratic forms $(x_1^2 + x_1x_2 + x_2^2) + c_1(x_3^2 + x_3x_4 + x_4^2) + c_2(x_5^2 + x_5x_6 + x_6^2) + c_3(x_7^2 + x_7x_8 + x_8^2)$, where $c_1, c_2, c_3 \in \{1, 2, 4, 8\}$, $c_1 \leq c_2 \leq c_3$.

1. INTRODUCTION

In this paper we consider the problem of finding the number of representations of the following quadratic forms in eight variables given by

$$(1) \quad a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 + b_1(x_5^2 + x_5x_6 + x_6^2) + b_2(x_7^2 + x_7x_8 + x_8^2),$$

where the coefficients $a_i \in \{1, 2, 3\}$, $1 \leq i \leq 4$ and $b_1, b_2 \in \{1, 2, 4\}$. Without loss of generality we can assume that $a_1 \leq a_2 \leq a_3 \leq a_4$ and $b_1 \leq b_2$. In [3], A. Alaca et. al considered similar types of quadratic forms in four variables, which are either sums of four squares with coefficients 1, 2, 3, 4 or 6 (7 such forms) or direct sum of the sums of two squares with coefficients 1 or 3 and the quadratic form $x^2 + xy + y^2$ with coefficients 1, 2 or 4 (6 such forms). They used theta function identities to determine the representation formulas for these 13 quadratic forms. In our recent work [16], we constructed bases for the space of modular forms of weight 4 for the group $\Gamma_0(48)$ with character, and used modular forms techniques to determine the number of representations of a natural number n by certain octonary quadratic forms with coefficients 1, 2, 3, 4, 6. Finding formulas for the number of representations for octonary quadratic forms with coefficients 1, 2, 3 or 6 were considered by various authors using several methods (see for example [1, 2, 4, 5, 6, 7]). In the present work, we adopt similar (modular forms) techniques to obtain the representation formulas. We show directly that the theta series corresponding to each of the quadratic form considered belongs to the space of modular forms of weight 4 on $\Gamma_0(24)$ with some character (depending on the coefficients). Now, by constructing a basis for the space of modular forms $M_4(\Gamma_0(24), \chi)$ we find the required formulas. Here χ is either the trivial Dirichlet character modulo 24 or one of the primitive Dirichlet characters (modulo m) $\chi_m = \left(\frac{m}{\cdot}\right)$, $m = 8, 12, 24$. Since $M_4(\Gamma_0(24), \chi) \subseteq M_4(\Gamma_0(48), \chi)$, where χ is a Dirichlet character modulo 24, we get the required explicit bases from the basis of modular forms $M_4(\Gamma_0(48), \psi)$, where ψ is a Dirichlet character modulo 48, which was constructed in [16].

In the second part of the paper, we consider the quadratic forms of eight variables given by:

$$(2) \quad (x_1^2 + x_1x_2 + x_2^2) + c_1(x_3^2 + x_3x_4 + x_4^2) + c_2(x_5^2 + x_5x_6 + x_6^2) + c_3(x_7^2 + x_7x_8 + x_8^2),$$

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where $c_1 \leq c_2 \leq c_3$ and $c_1, c_2, c_3 \in \{1, 2, 4, 8\}$. We note that for the c_i 's in the list, each of the quadratic form represents a theta series which belong to the space $M_4(\Gamma_0(24))$. Therefore, using our methods adopted for the earlier case, we also determine explicit formulas for the number of representations of a natural number by these class of quadratic forms.

The total number of such quadratic forms given by (1) with coefficients $a_i \in \{1, 2, 3\}$ and $b_i \in \{1, 2, 4\}$ is 90. Each quadratic form in this list is denoted as a sextuple $(a_1, a_2, a_3, a_4, b_1, b_2)$ and we list them in Table 1. We also put them in four classes corresponding to each of the modular forms space $M_4(\Gamma_0(24), \chi)$. Similarly, we list the quadratic forms (total 19) given by (2) in Table 2. In this case all the corresponding theta series belong to $M_4(\Gamma_0(24))$. We do not consider the case $(1, 1, 1, 1)$ as the formula is already known (see [20, Theorem 17.4]). It was shown that $s_8(n) = 24\sigma_3(n) + 216\sigma_3(n/3)$. In our notation (see §3) $s_8(n) = M(1, 1, 1, 1; n)$. Also, the cases $(1, 2, 2, 4)$ and $(1, 2, 4, 8)$ has been proved in [10] by using convolution sums method.

The paper is organized as follows. In §2 we present the theorems proved in this article and in §3 we give some preliminary results which are needed in proving the theorems. In §4 we give a proof of our theorems using the theory of modular forms.

Table 1.
List of quadratic forms in 8 variables given in (1)

$(a_1, a_2, a_3, a_4, b_1, b_2)$	space
$(1, 1, 1, 1, 1, 1), (1, 1, 1, 1, 1, 2), (1, 1, 1, 1, 1, 4), (1, 1, 1, 1, 2, 2)$ $(1, 1, 1, 1, 2, 4), (1, 1, 1, 1, 4, 4), (1, 1, 2, 2, 1, 1), (1, 1, 2, 2, 1, 2)$ $(1, 1, 2, 2, 1, 4), (1, 1, 2, 2, 2, 2), (1, 1, 2, 2, 2, 4), (1, 1, 2, 2, 4, 4)$ $(1, 1, 3, 3, 1, 1), (1, 1, 3, 3, 1, 2), (1, 1, 3, 3, 1, 4), (1, 1, 3, 3, 2, 2)$ $(1, 1, 3, 3, 2, 4), (1, 1, 3, 3, 4, 4), (2, 2, 2, 2, 1, 1), (2, 2, 2, 2, 1, 2)$ $(2, 2, 2, 2, 1, 4), (2, 2, 2, 2, 2, 2), (2, 2, 2, 2, 2, 4), (2, 2, 2, 2, 4, 4)$ $(2, 2, 3, 3, 1, 1), (2, 2, 3, 3, 1, 2), (2, 2, 3, 3, 1, 4), (2, 2, 3, 3, 2, 2)$ $(2, 2, 3, 3, 2, 4), (2, 2, 3, 3, 4, 4), (3, 3, 3, 3, 1, 1), (3, 3, 3, 3, 1, 2)$ $(3, 3, 3, 3, 1, 4), (3, 3, 3, 3, 2, 2), (3, 3, 3, 3, 2, 4), (3, 3, 3, 3, 4, 4)$	$M_4(\Gamma_0(24))$
$(1, 1, 1, 2, 1, 1), (1, 1, 1, 2, 1, 2), (1, 1, 1, 2, 1, 4), (1, 1, 1, 2, 2, 2)$ $(1, 1, 1, 2, 2, 4), (1, 1, 1, 2, 4, 4), (1, 2, 2, 2, 1, 1), (1, 2, 2, 2, 1, 2)$ $(1, 2, 2, 2, 1, 4), (1, 2, 2, 2, 2, 2), (1, 2, 2, 2, 2, 4), (1, 2, 2, 2, 4, 4)$ $(1, 2, 3, 3, 1, 1), (1, 2, 3, 3, 1, 2), (1, 2, 3, 3, 1, 4), (1, 2, 3, 3, 2, 2)$ $(1, 2, 3, 3, 2, 4), (1, 2, 3, 3, 4, 4)$	$M_4(\Gamma_0(24), \chi_8)$
$(1, 1, 1, 3, 1, 1), (1, 1, 1, 3, 1, 2), (1, 1, 1, 3, 1, 4), (1, 1, 1, 3, 2, 2)$ $(1, 1, 1, 3, 2, 4), (1, 1, 1, 3, 4, 4), (1, 2, 2, 2, 1, 1), (1, 2, 2, 3, 1, 2)$ $(1, 2, 2, 3, 1, 4), (1, 2, 2, 3, 2, 2), (1, 2, 2, 3, 2, 4), (1, 2, 2, 3, 4, 4)$ $(1, 3, 3, 3, 1, 1), (1, 3, 3, 3, 1, 2), (1, 3, 3, 3, 1, 4), (1, 3, 3, 3, 2, 2)$ $(1, 3, 3, 3, 2, 4), (1, 3, 3, 3, 4, 4)$	$M_4(\Gamma_0(24), \chi_{12})$
$(1, 1, 2, 3, 1, 1), (1, 1, 2, 3, 1, 2), (1, 1, 2, 3, 1, 4), (1, 1, 2, 3, 2, 2)$ $(1, 1, 2, 3, 2, 4), (1, 1, 2, 3, 4, 4), (2, 2, 2, 3, 1, 1), (2, 2, 2, 3, 1, 2)$ $(2, 2, 2, 3, 1, 4), (2, 2, 2, 3, 2, 2), (2, 2, 2, 3, 2, 4), (2, 2, 2, 3, 4, 4)$ $(2, 3, 3, 3, 1, 1), (2, 3, 3, 3, 1, 2), (2, 3, 3, 3, 1, 4), (2, 3, 3, 3, 2, 2)$ $(2, 3, 3, 3, 2, 4), (2, 3, 3, 3, 4, 4)$	$M_4(\Gamma_0(24), \chi_{24})$

Table 2.List of quadratic forms in (2) indicated by $(1, c_1, c_2, c_3)$.

$(1, c_1, c_2, c_3)$	space
$(1, 1, 1, 2), (1, 1, 1, 4), (1, 1, 1, 8), (1, 1, 2, 2), (1, 1, 2, 4), (1, 1, 2, 8), (1, 1, 4, 4)$ $(1, 1, 4, 8), (1, 1, 8, 8), (1, 2, 2, 2), (1, 2, 2, 4), (1, 2, 2, 8), (1, 2, 4, 4), (1, 2, 4, 8)$ $(1, 2, 8, 8), (1, 4, 4, 4), (1, 4, 4, 8), (1, 4, 8, 8), (1, 8, 8, 8)$	$M_4(\Gamma_0(24))$

2. STATEMENT OF RESULTS

Let \mathbb{N}, \mathbb{N}_0 and \mathbb{Z} denote the set of positive integers, non-negative integers and integers respectively. For $(a_1, a_2, a_3, a_4, b_1, b_2)$ as in Table 1, we define

$$N(a_1, a_2, a_3, a_4, b_1, b_2; n) :=$$

$$\# \left\{ (x_1, \dots, x_8) \in \mathbb{Z}^8 \mid n = \sum_{i=1}^4 a_i x_i^2 + b_1(x_5^2 + x_5 x_6 + x_6^2) + b_2(x_7^2 + x_7 x_8 + x_8^2) \right\}.$$

to be the number of representations of n by the quadratic form (1). Note that $N(a_1, a_2, a_3, a_4, b_1, b_2; 0) = 1$. The formulas corresponding to Table 1 are stated in the following theorem. Formulas are divided into four parts each corresponding to one of the four spaces of modular forms $M_4(\Gamma_0(24), \chi)$.

Theorem 2.1. *Let $n \in \mathbb{N}$.*

(i) *For each entry $(a_1, a_2, a_3, a_4, b_1, b_2)$ in Table 1 corresponding to the space $M_4(\Gamma_0(24))$, we have*

$$(3) \quad N(a_1, a_2, a_3, a_4, b_1, b_2; n) = \sum_{i=1}^{16} \alpha_i A_i(n),$$

where $A_i(n)$ are the Fourier coefficients of the basis elements f_i defined in §4.1 and the values of the constants α_i s are given in Table 3.

(ii) *For each entry $(a_1, a_2, a_3, a_4, b_1, b_2)$ in Table 1 corresponding to the space $M_4(\Gamma_0(24), \chi_8)$, we have*

$$(4) \quad N(a_1, a_2, a_3, a_4, b_1, b_2; n) = \sum_{i=1}^{14} \beta_i B_i(n),$$

where $B_i(n)$ are the Fourier coefficients of the basis elements g_i defined in §4.2 and the values of the constants β_i 's are given in Table 4.

(iii) *For each entry $(a_1, a_2, a_3, a_4, b_1, b_2)$ in Table 1 corresponding to the space $M_4(\Gamma_0(24), \chi_{12})$, we have*

$$(5) \quad N(a_1, a_2, a_3, a_4, b_1, b_2; n) = \sum_{i=1}^{16} \gamma_i C_i(n),$$

where $C_i(n)$ are the Fourier coefficients of the basis elements h_i defined in §4.3 and the values of the constants γ_i 's are given in Table 5.

(iv) *For each entry $(a_1, a_2, a_3, a_4, b_1, b_2)$ in Table 1 corresponding to the space $M_4(\Gamma_0(24), \chi_{24})$, we have*

$$(6) \quad N(a_1, a_2, a_3, a_4, b_1, b_2; n) = \sum_{i=1}^{14} \delta_i D_i(n),$$

where $D_i(n)$ are the Fourier coefficients of the basis elements F_i defined in §4.4 and the values of the constants δ_i 's are given in Table 6.

Now we consider the class of quadratic forms given by (2). For $(1, c_1, c_2, c_3)$ as in Table 2, we define

$$M(1, c_1, c_2, c_3; n) := \# \{ (x_1, \dots, x_8) \in \mathbb{Z}^8 \mid \\ n = (x_1^2 + x_1x_2 + x_2^2) + c_1(x_3^2 + x_3x_4 + x_4^2) + c_2(x_5^2 + x_5x_6 + x_6^2) + c_3(x_7^2 + x_7x_8 + x_8^2) \}.$$

to be the number of representations of n by the quadratic form (2). Note that $M(1, c_1, c_2, c_3; 0) = 1$. The formulas corresponding to Table 2 are stated in the following theorem.

Theorem 2.2. *Let $n \in \mathbb{N}$.*

For each entry $(1, c_1, c_2, c_3; n)$ in Table 2, we have

$$(7) \quad M(1, c_1, c_2, c_3; n) = \sum_{i=1}^{16} \nu_i A_i(n),$$

where $A_i(n)$ are the Fourier coefficients of the basis elements f_i defined in §4.1 and the values of the constants ν_i s are given in Table 7.

Remark 2.1. Since one can write down the exact formulas using the explicit Fourier coefficients of the basis elements and using the coefficients tables given in each of the cases, we have not stated explicit formulas in the theorems (due to large number of such formulas). However, in §5 (at the end of the Tables), we give some sample formulas corresponding to each case.

3. PRELIMINARIES

In this section we present some preliminary facts on modular forms. For $k \in \frac{1}{2}\mathbb{Z}$, let $M_k(\Gamma_0(N), \chi)$ denote the space of modular forms of weight k for the congruence subgroup $\Gamma_0(N)$ with character χ and $S_k(\Gamma_0(N), \chi)$ be the subspace of cusp forms of weight k for $\Gamma_0(N)$ with character χ . We assume $4|N$ when k is not an integer and in that case, the character χ (which is a Dirichlet character modulo N) is an even character. When χ is the trivial (principal) character modulo N , we shall denote the spaces by $M_k(\Gamma_0(N))$ and $S_k(\Gamma_0(N))$ respectively. Further, when $k \geq 4$ is an integer and $N = 1$, we shall denote these vector spaces by M_k and S_k respectively.

For an integer $k \geq 4$, let E_k denote the normalized Eisenstein series of weight k in M_k given by

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n,$$

where $q = e^{2i\pi z}$, $\sigma_r(n)$ is the sum of the r th powers of the positive divisors of n , and B_k is the k -th Bernoulli number defined by $\frac{x}{e^x - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} x^m$.

The classical theta function which is fundamental to the theory of modular forms of half-integral weight is defined by

$$(8) \quad \Theta(z) = \sum_{n \in \mathbb{Z}} q^{n^2},$$

and is a modular form in the space $M_{1/2}(\Gamma_0(4))$. Another function which is mainly used in our work is the Dedekind eta function $\eta(z)$ and it is given by

$$(9) \quad \eta(z) = q^{1/24} \prod_{n \geq 1} (1 - q^n).$$

An eta-quotient is a finite product of integer powers of $\eta(z)$ and we denote it as follows:

$$(10) \quad \prod_{i=1}^s \eta^{r_i}(d_i z) := d_1^{r_1} d_2^{r_2} \cdots d_s^{r_s},$$

where d_i 's are positive integers and r_i 's are non-zero integers.

We denote the theta series associated to the quadratic form $x^2 + xy + y^2$ by

$$(11) \quad \mathcal{F}(z) = \sum_{x,y \in \mathbb{Z}} q^{x^2 + xy + y^2}.$$

This function is referred to as the Borweins' two dimensional theta function in the literature. By [17, Theorem 4], it follows that $\mathcal{F}(z)$ is a modular form in $M_1(\Gamma_0(3), \chi_{-3})$. Here and in the sequel, for $m < 0$, the character χ_m is the odd Dirichlet character modulo $|m|$ given by $\left(\frac{-m}{\cdot}\right)$.

In the following we shall present some facts about modular forms of integral and half-integral weights, which we shall be using in our proof. We state them as lemmas, whose proofs follow from elementary theory of modular forms (of integral and half-integral weights).

Lemma 1. (*Duplication of modular forms*)

If f is a modular form in $M_k(\Gamma_0(N), \chi)$, then for a positive integer d , the function $f(dz)$ is a modular form in $M_k(\Gamma_0(dN), \chi)$, if k is an integer and it belongs to the space $M_k(\Gamma_0(dN), \chi\chi_d)$, if k is a half-integer.

Lemma 2. *For positive integers r, r_1, r_2, d_1, d_2 , we have*

$$(12) \quad \Theta^r(d_1 z) \in \begin{cases} M_{r/2}(\Gamma_0(4d_1), \chi_{d_1}) & \text{if } r \text{ is odd,} \\ M_{r/2}(\Gamma_0(4d_1), \chi_{-4}) & \text{if } r \equiv 2 \pmod{4}, \\ M_{r/2}(\Gamma_0(4d_1)) & \text{if } r \equiv 0 \pmod{4}. \end{cases}$$

(13)

$$\Theta^{r_1}(d_1 z) \cdot \Theta^{r_2}(d_2 z) \in \begin{cases} M_{\frac{r_1+r_2}{2}}(\Gamma_0(4[d_1, d_2]), \chi_{(-d_1 d_2)}) & \text{if } r_1 r_2 \text{ is odd, } r_1 + r_2 \equiv 2 \pmod{4}, \\ M_{\frac{r_1+r_2}{2}}(\Gamma_0(4[d_1, d_2]), \chi_{(d_1 d_2)}) & \text{if } r_1 r_2 \text{ is odd, } r_1 + r_2 \equiv 0 \pmod{4}. \end{cases}$$

Lemma 3. *If $f_i \in M_{k_i}(\Gamma_0(M_i), \psi_i)$, $i = 1, 2$, then the product $f_1 \cdot f_2$ is a modular form in $M_{k_1+k_2}(\Gamma_0(M), \psi_1 \psi_2)$, where $M = \text{lcm}(M_1, M_2)$.*

Lemma 4. *The vector space $M_k(\Gamma_1(N))$ is decomposed as a direct sum:*

$$(14) \quad M_k(\Gamma_1(N)) = \oplus_{\chi} M_k(\Gamma_0(N), \chi),$$

where the direct sum varies over all Dirichlet characters modulo N if the weight k is a positive integer and varies over all even Dirichlet characters modulo $N, 4|N$, if the weight k is half-integer. Further, if k is an integer, one has $M_k(\Gamma_0(N), \chi) = \{0\}$, if $\chi(-1) \neq (-1)^k$. We also have the following decomposition of the space into subspaces of Eisenstein series and cusp forms:

$$(15) \quad M_k(\Gamma_0(N), \chi) = \mathcal{E}_k(\Gamma_0(N), \chi) \oplus S_k(\Gamma_0(N), \chi),$$

where $\mathcal{E}_k(\Gamma_0(N), \chi)$ is the space generated by the Eisenstein series of weight k on $\Gamma_0(N)$ with character χ .

Lemma 5. *By the Atkin-Lehner theory of newforms, the space $S_k(\Gamma_0(N), \chi)$ can be decomposed into the space of newforms and oldforms:*

$$(16) \quad S_k(\Gamma_0(N), \chi) = S_k^{new}(\Gamma_0(N), \chi) \oplus S_k^{old}(\Gamma_0(N), \chi),$$

where the above is an orthogonal direct sum (with respect to the Petersson scalar product) and

$$(17) \quad S_k^{old}(\Gamma_0(N), \chi) = \bigoplus_{\substack{r|N, r < N \\ rd|N}} S_k^{new}(\Gamma_0(r), \chi) | B(d).$$

In the above, $S_k^{new}(\Gamma_0(N), \chi)$ is the space of newforms and $S_k^{old}(\Gamma_0(N), \chi)$ is the space of oldforms and the operator $B(d)$ is given by $f(z) \mapsto f(dz)$.

Lemma 6. *Suppose that χ and ψ are primitive Dirichlet characters with conductors M and N , respectively. For a positive integer k , let*

$$(18) \quad E_{k, \chi, \psi}(z) := c_0 + \sum_{n \geq 1} \left(\sum_{d|n} \psi(d) \cdot \chi(n/d) d^{k-1} \right) q^n,$$

where

$$c_0 = \begin{cases} 0 & \text{if } M > 1, \\ -\frac{B_{k, \psi}}{2k} & \text{if } M = 1, \end{cases}$$

and $B_{k, \psi}$ denotes generalized Bernoulli number with respect to the character ψ . Then, the Eisenstein series $E_{k, \chi, \psi}(z)$ belongs to the space $M_k(\Gamma_0(MN), \chi/\psi)$, provided $\chi(-1)\psi(-1) = (-1)^k$ and $MN \neq 1$. When $\chi = \psi = 1$ (i.e., when $M = N = 1$) and $k \geq 4$, we have $E_{k, \chi, \psi}(z) = -\frac{B_k}{2k} E_k(z)$, where E_k is the normalized Eisenstein series of integer weight k as defined before. We refer to [15, 19] for details.

We give a notation to the inner sum in (18):

$$(19) \quad \sigma_{k-1; \chi, \psi}(n) := \sum_{d|n} \psi(d) \cdot \chi(n/d) d^{k-1}.$$

For more details on the theory of modular forms of integral and half-integral weights, we refer to [8, 9, 11, 15, 17, 18].

4. PROOFS OF THEOREMS

In this section, we shall give a proof of our results. As mentioned in the introduction, we shall be using the theory of modular forms.

The basic functions for the two types of quadratic forms considered in this paper are $\Theta(z)$ and $\mathcal{F}(z)$. To each quadratic form in (1) with coefficients $(a_1, a_2, a_3, a_4, b_1, b_2)$ as in Table 1, the associated theta series is given by

$$\Theta(a_1 z) \Theta(a_2 z) \Theta(a_3 z) \Theta(a_4 z) \mathcal{F}(b_1 z) \mathcal{F}(b_2 z).$$

Using Lemma 1 and 2 along with the fact that $\mathcal{F}(z) \in M_1(\Gamma_0(3), \chi_{-3})$, it follows that the above product is a modular form in $M_4(\Gamma_0(24), \chi)$, where the character χ is one of the four characters that appear in Table 1 and it is determined by the coefficients a_1, a_2, a_3, a_4 . As remarked earlier, the theta series corresponding to the form $x^2 + xy + y^2$ is given by

(11) and it belongs to the space $M_1(\Gamma_0(3), \chi_{-3})$. Therefore, the associated modular form corresponding to the quadratic forms defined by (2) is given explicitly by

$$\mathcal{F}(z)\mathcal{F}(c_1z)\mathcal{F}(c_2z)\mathcal{F}(c_3z).$$

Again by using Lemmas 1, 2 and 3 it follows that the above product is a modular form in $M_4(\Gamma_0(24))$. Therefore, in order to get the required formulae for $N(a_1, a_2, a_3, a_4, b_1, b_2; n)$ and $M(1, c_1, c_2, c_3; n)$ we need a basis for the above spaces of modular forms of level 24. (We have used the L -functions and modular forms database [12] and [14] to get some of the cusp forms of weight 4.)

4.1. A basis for $M_4(\Gamma_0(24))$ and proof of Theorem 2.1(i). The vector space $M_4(\Gamma_0(24))$ has dimension 16 and we have $\dim_{\mathbb{C}} \mathcal{E}_4(\Gamma_0(24)) = 8$ and $\dim_{\mathbb{C}} S_4(\Gamma_0(24)) = 8$. For $d = 6, 8, 12$ and 24, $S_4^{new}(\Gamma_0(d))$ is one-dimensional. Let us define some eta-quotients and use them to give an explicit basis for $S_4(\Gamma_0(24))$. Let

$$(20) \quad f_{4,6}(z) = 1^2 2^2 3^2 6^2 := \sum_{n \geq 1} a_{4,6}(n) q^n, \quad f_{4,8}(z) = 2^4 4^4 := \sum_{n \geq 1} a_{4,8}(n) q^n,$$

$$(21) \quad f_{4,12}(z) = 1^{-1} 2^2 3^3 4^3 6^2 12^{-1} - 1^3 2^2 3^{-1} 4^{-1} 6^2 12^3 := \sum_{n \geq 1} a_{4,12}(n) q^n,$$

$$(22) \quad f_{4,24}(z) = 1^{-4} 2^{11} 3^{-4} 4^{-3} 6^{11} 12^{-3} := \sum_{n \geq 1} a_{4,24}(n) q^n.$$

We use the following notation in the sequel. For a Dirichlet character χ and a function f with Fourier expansion $f(z) = \sum_{n \geq 1} a(n) q^n$, we define the twisted function $f \otimes \chi(z)$ as follows.

$$(23) \quad f \otimes \chi(z) = \sum_{n \geq 1} \chi(n) a(n) q^n.$$

A basis for the space $M_4(\Gamma_0(24))$ is given in the following proposition.

Proposition 4.1. *A basis for the Eisenstein series space $\mathcal{E}_4(\Gamma_0(24))$ is given by*

$$(24) \quad \{E_4(tz), t|24\}$$

and a basis for the space of cusp forms $S_4(\Gamma_0(24))$ is given by

$$(25) \quad \{f_{4,6}(t_1z), t_1|4; f_{4,8}(t_2z), t_2|3; f_{4,12}(t_3z), t_3|2; f_{4,24} \otimes \chi_4(z)\}$$

Together they form a basis for $M_4(\Gamma_0(24))$.

For the sake of simplicity in the formulae, we list these basis elements as $\{f_i(z) | 1 \leq i \leq 16\}$, where $f_1(z) = E_4(z)$, $f_2(z) = E_4(2z)$, $f_3(z) = E_4(3z)$, $f_4(z) = E_4(4z)$, $f_5(z) = E_4(6z)$, $f_6(z) = E_4(8z)$, $f_7(z) = E_4(12z)$, $f_8(z) = E_4(24z)$, $f_9(z) = f_{4,6}(z)$, $f_{10}(z) = f_{4,6}(2z)$, $f_{11}(z) = f_{4,6}(4z)$, $f_{12}(z) = f_{4,8}(z)$, $f_{13}(z) = f_{4,8}(3z)$, $f_{14}(z) = f_{4,12}(z)$, $f_{15}(z) = f_{4,12}(2z)$, $f_{16}(z) = f_{4,24} \otimes \chi_4(z)$

For $1 \leq i \leq 16$, we denote the Fourier coefficients of the basis functions $f_i(z)$ as

$$f_i(z) = \sum_{n \geq 1} A_i(n) q^n.$$

We are now ready to prove the theorem. Noting that all the 36 cases corresponding to the trivial character in Table 1, the resulting functions belong to the space of modular forms of weight 4 on $\Gamma_0(24)$ with trivial character (using Lemmas 1 to 3). So, we can

express these theta functions as a linear combination of the basis given in Proposition 4.1 as follows.

$$(26) \quad \Theta(a_1 z) \Theta(a_2 z) \Theta(a_3 z) \Theta(a_4 z) \mathcal{F}(b_1 z) \mathcal{F}(b_2 z) = \sum_{i=1}^{16} \alpha_i f_i(z),$$

where α_i 's are some explicit constants. Comparing the n -th Fourier coefficients on both the sides, we get

$$N(a_1, a_2, a_3, a_4, b_1, b_2; n) = \sum_{i=1}^{16} \alpha_i A_i(n).$$

Explicit values for the constants α_i , $1 \leq i \leq 16$ corresponding to these 36 cases are given in Table 3.

4.2. A basis for $M_4(\Gamma_0(24), \chi_8)$ and proof of Theorem 2.1(ii). The vector space $M_4(\Gamma_0(24), \chi_8)$ has dimension 14 and we have $\dim_{\mathbb{C}} \mathcal{E}_4(\Gamma_0(24), \chi_8) = 4$ and $\dim_{\mathbb{C}} S_4(\Gamma_0(24), \chi_8) = 10$. For $d = 6$ and 12 , $S_4^{new}(\Gamma_0(d), \chi_8) = \{0\}$. Also $S_4^{new}(\Gamma_0(8), \chi_8)$ is 2-dimensional and $S_4^{new}(\Gamma_0(24), \chi_8)$ is 6-dimensional.

In order to give explicit basis for this space, we define the following

$$(27) \quad E_{4,1,\chi_8}(z) = \frac{11}{2} + \sum_{n \geq 1} \sigma_{3;1,\chi_8}(n) q^n, \quad E_{4,\chi_8,1}(z) = \sum_{n \geq 1} \sigma_{3;\chi_8,1}(n) q^n.$$

$$(28) \quad f_{4,8,\chi_8;1}(z) = 1^{-2} 2^{11} 4^{-3} 8^2 = \sum_{n \geq 1} a_{4,8,\chi_8;1}(n) q^n, \quad f_{4,8,\chi_8;2}(z) = 1^2 2^{-3} 4^{11} 8^{-2} = \sum_{n \geq 1} a_{4,8,\chi_8;2}(n) q^n.$$

For the space of Eisenstein series we use the basis elements of $\mathcal{E}_4(\Gamma_0(8), \chi_8)$ given in (27). A basis for $S_4^{new}(\Gamma_0(8), \chi_8)$ is given in (28). The following six eta-quotients span the space $S_4^{new}(\Gamma_0(24), \chi_8)$.

$$(29) \quad \begin{aligned} f_{4,24,\chi_8;1}(z) &= 1^2 2^1 3^{-4} 4^1 6^{10} 8^2 12^{-4} := \sum_{n \geq 1} a_{4,24,\chi_8;1}(n) q^n, \\ f_{4,24,\chi_8;2}(z) &= 1^1 2^3 3^{-1} 4^1 6^4 8^{-1} 24^1 := \sum_{n \geq 1} a_{4,24,\chi_8;2}(n) q^n, \\ f_{4,24,\chi_8;3}(z) &= 1^{-1} 2^4 3^1 6^3 8^1 12^1 24^{-1} := \sum_{n \geq 1} a_{4,24,\chi_8;3}(n) q^n, \\ f_{4,24,\chi_8;4}(z) &= 1^{-2} 2^4 4^2 6^1 8^2 12^1 := \sum_{n \geq 1} a_{4,24,\chi_8;4}(n) q^n, \\ f_{4,24,\chi_8;5}(z) &= 2^1 3^{-2} 4^1 6^4 12^2 24^2 := \sum_{n \geq 1} a_{4,24,\chi_8;5}(n) q^n, \\ f_{4,24,\chi_8;6}(z) &= 1^{-6} 2^{14} 6^1 8^{-2} 12^1 := \sum_{n \geq 1} a_{4,24,\chi_8;6}(n) q^n \end{aligned}$$

A basis for the space $M_4(\Gamma_0(24), \chi_8)$ is given in the following proposition.

Proposition 4.2. *A basis for the space $M_4(\Gamma_0(24), \chi_8)$ is given by*

$$(30) \quad \{E_{4,1,\chi_8}(tz), E_{4,\chi_8,1}(tz), t|3; f_{4,8,\chi_8;1}(t_1 z), f_{4,8,\chi_8;2}(t_1 z), t_1|3; f_{4,24,\chi_8;1}(z), f_{4,24,\chi_8;2}(z), f_{4,24,\chi_8;3}(z), f_{4,24,\chi_8;4}(z), f_{4,24,\chi_8;5}(z), f_{4,24,\chi_8;6}(z)\}$$

where $E_{4,1,\chi_8}(z)$ and $E_{4,\chi_8,1}(z)$ are defined in (27), $f_{4,8,\chi_8;i}(z)$, $i = 1, 2$ are defined in (28) and $f_{4,24,\chi_8;j}(z)$, $1 \leq j \leq 6$ are defined by (29).

For the sake of simplifying the notation, we shall list the basis in Proposition 4.2 as

$$g_i(z) = \sum_{n \geq 1} B_i(n) q^n, \quad 1 \leq i \leq 14,$$

where $g_1(z) = E_{4,1,\chi_8}(z)$, $g_2(z) = E_{4,1,\chi_8}(3z)$, $g_3(z) = E_{4,\chi_8,1}(z)$, $g_4(z) = E_{4,\chi_8,1}(3z)$, $g_5(z) = f_{4,8,\chi_8;1}(z)$, $g_6(z) = f_{4,8,\chi_8;1}(3z)$, $g_7(z) = f_{4,8,\chi_8;2}(z)$, $g_8(z) = f_{4,8,\chi_8;2}(3z)$, $g_9(z) = f_{4,24,\chi_8;1}(z)$, $g_{10}(z) = f_{4,24,\chi_8;2}(z)$, $g_{11}(z) = f_{4,24,\chi_8;3}(z)$, $g_{12}(z) = f_{4,24,\chi_8;4}(z)$, $g_{13}(z) = f_{4,24,\chi_8;5}(z)$, $g_{14}(z) = f_{4,24,\chi_8;6}(z)$,

We now prove Theorem 2.1(ii). In this case, for all the 18 sextuples corresponding to the χ_8 character space (in Table 1), the resulting products of theta functions are modular forms of weight 4 on $\Gamma_0(24)$ with character χ_8 (By Lemma 1 to 3). So, we can express these products of theta functions as a linear combination of the basis given in Proposition 4.2:

$$(31) \quad \Theta(a_1 z) \Theta(a_2 z) \Theta(a_3 z) \Theta(a_4 z) \mathcal{F}(b_1 z) \mathcal{F}(b_2 z) = \sum_{i=1}^{14} \beta_i g_i(z).$$

Comparing the n -th Fourier coefficients on both the sides, we get

$$N(a_1, a_2, a_3, a_4, b_1, b_2; n) = \sum_{i=1}^{14} \beta_i B_i(n).$$

Explicit values for the constants β_i , $1 \leq i \leq 14$ corresponding to these 18 cases are given in Table 4.

4.3. A basis for $M_4(\Gamma_0(24), \chi_{12})$ and proof of Theorem 2.1(iii). The dimension of the space in this case is 16, with $\dim_{\mathbb{C}} \mathcal{E}_4(\Gamma_0(24), \chi_{12}) = 8$ and $\dim_{\mathbb{C}} S_4(\Gamma_0(24), \chi_{12}) = 8$. The old class is spanned by the space $S_4^{new}(\Gamma_0(12), \chi_{12})$, which is 4 dimensional with spanning functions given by the following four eta-quotients:

$$(32) \quad \begin{aligned} f_{4,12,\chi_{12};1}(z) &= 2^{-1} 3^4 4^2 6^5 12^{-2}, & f_{4,12,\chi_{12};2}(z) &= 3^4 4^3 6^{-2} 12^3, \\ f_{4,12,\chi_{12};3}(z) &= 2^2 3^4 4^{-1} 6^{-4} 12^7, & f_{4,12,\chi_{12};4}(z) &= 1^4 4^{-1} 6^{-2} 12^7. \end{aligned}$$

We write the Fourier expansions of these forms as $f_{4,12,\chi_{12};j}(z) = \sum_{n \geq 1} a_{4,12,\chi_{12};j}(n) q^n$, $1 \leq j \leq 4$.

In the following proposition we give a basis for the space $M_4(\Gamma_0(24), \chi_{12})$.

Proposition 4.3. *A basis for the space $M_4(\Gamma_0(24), \chi_{12})$ is given by*

$$(33) \quad \{E_{4,1,\chi_{12}}(tz), E_{4,\chi_{12},1}(tz), E_{4,\chi_{-4},\chi_{-3}}(tz), E_{4,\chi_{-3},\chi_{-4}}(tz), t|2; f_{4,12,\chi_{12};j}(t_1 z), t_1|2, 1 \leq j \leq 4\},$$

where the Eisenstein series in the basis are defined by (18).

Let us denote the 16 basis elements in the above proposition as follows. $\{h_i(z) | 1 \leq i \leq 16\}$, where $h_1(z) = E_{4,1,\chi_{12}}(z)$, $h_2(z) = E_{4,\chi_{12},1}(z)$, $h_3(z) = E_{4,\chi_{-4},\chi_{-3}}(z)$, $h_4(z) = E_{4,\chi_{-3},\chi_{-4}}(z)$, $h_5(z) = E_{4,1,\chi_{12}}(2z)$, $h_6(z) = E_{4,\chi_{12},1}(2z)$, $h_7(z) = E_{4,\chi_{-4},\chi_{-3}}(2z)$, $h_8(z) = E_{4,\chi_{-3},\chi_{-4}}(2z)$, $h_{8+j}(z) = f_{4,12,\chi_{12};j}(z)$, $1 \leq j \leq 4$, $h_{12+j}(z) = f_{4,12,\chi_{12};j}(2z)$, $1 \leq j \leq 4$.

To prove Theorem 2.1(iii), we consider the case of 18 sextuples corresponding to the character χ_{12} in Table 1. The resulting products of theta functions are modular forms of weight 4 on $\Gamma_0(24)$ with character χ_{12} (once again we use Lemmas 1 to 3 to get this). So,

we can express each of these products of theta functions as a linear combination of the basis given in Proposition 4.3 as follows.

$$(34) \quad \Theta(a_1 z) \Theta(a_2 z) \Theta(a_3 z) \Theta(a_4 z) \mathcal{F}(b_1 z) \mathcal{F}(b_2 z) = \sum_{i=1}^{16} \gamma_i g_i(z).$$

Comparing the n -th Fourier coefficients on both the sides, we get

$$N(a_1, a_2, a_3, a_4, b_1, b_2; n) = \sum_{i=1}^{16} \gamma_i C_i(n).$$

Explicit values of the constants γ_i , $1 \leq i \leq 16$ corresponding to these 18 cases are given in Table 5.

4.4. A basis for $M_4(\Gamma_0(24), \chi_{24})$ and proof of Theorem 2.1(iv). We have $\dim_{\mathbb{C}} M_4(\Gamma_0(24), \chi_{24}) = 14$ and $\dim_{\mathbb{C}} \mathcal{E}_4(\Gamma_0(24), \chi_{24}) = 4$. To get the span of the Eisenstein series space $\mathcal{E}_4(\Gamma_0(24), \chi_{24})$, we use the Eisenstein series $E_{4,\chi,\psi}(z)$ defined in (18), where $\chi, \psi \in \{1, \chi_{-8}, \chi_{-12}, \chi_{24}\}$. Note that for $d = 6, 8$ and 12 , $S_4^{new}(\Gamma_0(d), \chi_{24}) = \{0\}$ and the space $S_4^{new}(\Gamma_0(24), \chi_{24})$ is spanned by the following ten eta-quotients (notation as in (10)):

$$(35) \quad \begin{aligned} f_{4,24,\chi_{24};1}(z) &= 3^{-2} 6^7 8^3 12^3 24^{-3}, & f_{4,24,\chi_{24};2}(z) &= 3^2 4^7 6^{-3} 8^{-2} 12^4, \\ f_{4,24,\chi_{24};3}(z) &= 3^2 4^{-3} 6^1 8^6 12^2, & f_{4,24,\chi_{24};4}(z) &= 3^2 6^{-3} 8^3 12^5 24^1, \\ f_{4,24,\chi_{24};5}(z) &= 3^2 4^2 6^{-3} 8^{-1} 12^3 24^5, & f_{4,24,\chi_{24};6}(z) &= 3^2 4^1 6^1 8^{-2} 12^{-2} 24^8, \\ f_{4,24,\chi_{24};7}(z) &= 3^2 4^1 6^1 8^{-2} 12^{-2} 24^8, & f_{4,24,\chi_{24};8}(z) &= 1^1 3^{-1} 6^1 8^{-2} 12^1 24^8, \\ f_{4,24,\chi_{24};9}(z) &= 2^2 3^6 4^1 6^{-3} 8^2, & f_{4,24,\chi_{24};10}(z) &= 3^2 4^3 6^5 12^{-4} 24^2. \end{aligned}$$

We write the Fourier expansions as $f_{4,24,\chi_{24};j}(z) = \sum_{n \geq 1} a_{4,24,\chi_{24};j}(n) q^n$. We now give a basis for the space $M_4(\Gamma_0(24), \chi_{24})$ in the following proposition.

Proposition 4.4. *The following functions span the space $M_4(\Gamma_0(24), \chi_{24})$.*

$$(36) \quad \{E_{4,1,\chi_{24}}(z), E_{4,\chi_{24},1}(z), E_{4,\chi_{-8},\chi_{-3}}(z), E_{4,\chi_{-3},\chi_{-8}}(z), f_{4,24,\chi_{24};j}(z), 1 \leq j \leq 10; \}.$$

We list these basis elements as $\{F_i(z) | 1 \leq i \leq 14\}$, where $F_1(z) = E_{4,1,\chi_{24}}(z)$, $F_2(z) = E_{4,\chi_{24},1}(z)$, $F_3(z) = E_{4,\chi_{-8},\chi_{-3}}(z)$, $F_4(z) = E_{4,\chi_{-3},\chi_{-8}}(z)$, $F_5(z) = f_{4,24,\chi_{24};1}(z)$, $F_6(z) = f_{4,24,\chi_{24};2}(z)$, $F_7(z) = f_{4,24,\chi_{24};3}(z)$, $F_8(z) = f_{4,24,\chi_{24};4}(z)$, $F_9(z) = f_{4,24,\chi_{24};5}(z)$, $F_{10}(z) = f_{4,24,\chi_{24};6}(z)$, $F_{11}(z) = f_{4,24,\chi_{24};7}(z)$, $F_{12}(z) = f_{4,24,\chi_{24};8}(z)$, $F_{13}(z) = f_{4,24,\chi_{24};9}(z)$, $F_{14}(z) = f_{4,24,\chi_{24};10}(z)$,

As in the previous cases, we denote the Fourier coefficients of these basis functions by

$$F_i(z) = \sum_{n \geq 1} D_i(n) q^n, \quad 1 \leq i \leq 14.$$

To get the formula in Theorem 2.1(iv), we note that for all the 18 sextuples corresponding to the character χ_{24} in Table 1, the resulting functions belong to the space $M_4(\Gamma_0(24), \chi_{24})$, by using Lemmas 1 to 3. So, as before, we express these theta functions as linear combinations of the basis elements:

$$(37) \quad \Theta(a_1 z) \Theta(a_2 z) \Theta(a_3 z) \Theta(a_4 z) \mathcal{F}(b_1 z) \mathcal{F}(b_2 z) = \sum_{i=1}^{14} \delta_i g_i(z).$$

Comparing the n -th Fourier coefficients on both the sides, we get

$$N(a_1, a_2, a_3, a_4, b_1, b_2; n) = \sum_{i=1}^{14} \delta_i D_i(n).$$

Explicit values of the constants δ_i , $1 \leq i \leq 14$ corresponding to these 18 cases corresponding to character χ_{24} are given in Table 6.

4.5. Proof of Theorem 2.2. This theorem is corresponding to Table 2 and in this case all the product functions

$$\mathcal{F}(z)\mathcal{F}(c_1z)\mathcal{F}(c_2z)\mathcal{F}(c_3z)$$

belong to the space $M_4(\Gamma_0(24))$. Therefore, proceeding as in the proof of Theorem 2.1(i), we express these theta functions as linear combinations of the basis elements:

$$(38) \quad \mathcal{F}(z)\mathcal{F}(c_1z)\mathcal{F}(c_2z)\mathcal{F}(c_3z) = \sum_{i=1}^{16} \nu_i f_i(z).$$

Comparing the n -th Fourier coefficients on both the sides, we get

$$M(1, c_1, c_2, c_3; n) = \sum_{i=1}^{16} \nu_i A_i(n).$$

The constants ν_i , $1 \leq i \leq 16$ corresponding to the 19 cases of table 2 are given in Table 7.

5. LIST OF TABLES AND SAMPLE FORMULAS

In this section we list the remaining tables mentioned in the theorems and provide explicit sample formulas in some cases. In the first subsection we list the tables and in the second subsection we give the sample formulas.

5.1. List of tables. In this section, we list the tables 3, 4, 5, 6 and 7 which give the explicit coefficients that appear in the formulas of Theorem 2.1 and Theorem 2.2.

Table 3. (Theorem 2.1 (i))

$(a_1 a_2 a_3 a_4, b_1 b_2)$	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}	α_{14}	α_{15}	α_{16}
(1111, 11)	$\frac{7}{75}$	$-\frac{7}{100}$	$-\frac{9}{25}$	$-\frac{28}{75}$	$\frac{27}{100}$	0	$\frac{36}{25}$	0	$-\frac{72}{5}$	$-\frac{288}{5}$	0	0	0	12	0	0
(1111, 12)	$\frac{13}{300}$	$-\frac{13}{200}$	$\frac{9}{100}$	$\frac{26}{75}$	$-\frac{27}{200}$	0	$\frac{18}{25}$	0	$\frac{48}{5}$	$\frac{96}{5}$	0	0	0	-6	0	0
(1111, 14)	$\frac{7}{300}$	0	$-\frac{9}{100}$	$-\frac{28}{75}$	0	0	$\frac{36}{25}$	0	$-\frac{18}{5}$	$\frac{72}{5}$	0	0	0	12	0	0
(1111, 22)	$\frac{7}{300}$	0	$-\frac{9}{100}$	$-\frac{28}{75}$	0	0	$\frac{36}{25}$	0	$\frac{12}{5}$	$-\frac{48}{5}$	0	0	0	0	0	0
(1111, 24)	$\frac{13}{1200}$	$-\frac{13}{400}$	$\frac{9}{400}$	$\frac{26}{75}$	$-\frac{27}{400}$	0	$\frac{18}{25}$	0	$\frac{12}{5}$	$\frac{96}{5}$	0	0	0	3	0	0
(1111, 44)	$\frac{7}{1200}$	$\frac{7}{400}$	$-\frac{9}{400}$	$-\frac{28}{75}$	$-\frac{27}{400}$	0	$\frac{36}{25}$	0	$\frac{18}{5}$	$\frac{72}{5}$	0	0	0	3	0	0
(1122, 11)	$\frac{7}{150}$	$\frac{7}{150}$	$-\frac{9}{50}$	$\frac{7}{300}$	$\frac{9}{50}$	$-\frac{28}{75}$	$-\frac{9}{100}$	$\frac{36}{25}$	$-\frac{36}{5}$	-48	$-\frac{768}{5}$	-3	-81	6	36	9
(1122, 12)	$\frac{13}{600}$	$-\frac{13}{600}$	$\frac{9}{200}$	$-\frac{13}{600}$	$-\frac{9}{200}$	$\frac{26}{75}$	$-\frac{9}{200}$	$\frac{18}{25}$	$\frac{24}{5}$	12	$\frac{96}{5}$	$\frac{15}{2}$	$\frac{81}{2}$	-3	-6	$-\frac{9}{2}$
(1122, 14)	$\frac{7}{600}$	$-\frac{7}{600}$	$-\frac{9}{200}$	$\frac{7}{300}$	$\frac{9}{200}$	$-\frac{28}{75}$	$-\frac{9}{100}$	$\frac{36}{25}$	$-\frac{9}{5}$	6	$-\frac{48}{5}$	$-\frac{3}{2}$	$-\frac{81}{2}$	6	0	$\frac{9}{2}$
(1122, 22)	$\frac{7}{600}$	$-\frac{7}{600}$	$-\frac{9}{200}$	$\frac{7}{300}$	$\frac{9}{200}$	$-\frac{28}{75}$	$-\frac{9}{100}$	$\frac{36}{25}$	$\frac{6}{5}$	-12	$-\frac{288}{5}$	0	0	0	12	0
(1122, 24)	$\frac{13}{2400}$	$-\frac{13}{2400}$	$\frac{9}{800}$	$-\frac{13}{600}$	$-\frac{9}{800}$	$\frac{26}{75}$	$-\frac{9}{200}$	$\frac{18}{25}$	$\frac{6}{5}$	12	$\frac{96}{5}$	0	0	$\frac{3}{2}$	-6	0
(1122, 44)	$\frac{7}{2400}$	$-\frac{7}{2400}$	$-\frac{9}{800}$	$\frac{7}{300}$	$\frac{9}{800}$	$-\frac{28}{75}$	$-\frac{9}{100}$	$\frac{36}{25}$	$\frac{9}{5}$	6	$-\frac{48}{5}$	0	0	$\frac{3}{2}$	0	0
(1133, 11)	$\frac{2}{75}$	$-\frac{1}{30}$	$\frac{6}{25}$	$\frac{8}{75}$	$-\frac{3}{10}$	0	$\frac{24}{25}$	0	$\frac{48}{5}$	$\frac{288}{5}$	0	0	0	0	0	0
(1133, 12)	$\frac{1}{60}$	$-\frac{1}{120}$	$-\frac{3}{20}$	$-\frac{2}{15}$	$\frac{3}{40}$	0	$\frac{6}{5}$	0	0	0	0	0	0	6	0	0
(1133, 14)	$\frac{1}{150}$	$-\frac{1}{75}$	$\frac{3}{50}$	$\frac{8}{75}$	$-\frac{3}{25}$	0	$\frac{24}{25}$	0	$\frac{42}{5}$	$\frac{168}{5}$	0	0	0	0	0	0
(1133, 22)	$\frac{1}{150}$	$-\frac{1}{75}$	$\frac{3}{50}$	$\frac{8}{75}$	$-\frac{3}{25}$	0	$\frac{24}{25}$	0	$\frac{12}{5}$	$\frac{48}{5}$	0	0	0	0	0	0
(1133, 24)	$\frac{1}{240}$	$-\frac{1}{240}$	$-\frac{3}{80}$	$-\frac{2}{15}$	$-\frac{3}{80}$	0	$\frac{6}{5}$	0	0	0	0	0	0	3	0	0
(1133, 44)	$\frac{1}{600}$	$-\frac{1}{120}$	$\frac{3}{200}$	$\frac{8}{75}$	$-\frac{3}{40}$	0	$\frac{24}{25}$	0	$\frac{18}{5}$	$\frac{48}{5}$	0	0	0	0	0	0
(2222, 11)	$\frac{7}{400}$	$\frac{91}{1200}$	$-\frac{27}{400}$	$-\frac{7}{100}$	$-\frac{117}{400}$	$-\frac{28}{75}$	$\frac{27}{100}$	$\frac{36}{25}$	$-\frac{36}{5}$	$-\frac{312}{5}$	$-\frac{768}{5}$	-3	-81	9	36	9
(2222, 12)	$\frac{13}{800}$	$-\frac{247}{2400}$	$\frac{27}{800}$	$\frac{13}{200}$	$-\frac{171}{800}$	$\frac{26}{75}$	$\frac{27}{200}$	$\frac{18}{25}$	$\frac{18}{5}$	$\frac{84}{5}$	$\frac{96}{5}$	$\frac{15}{2}$	$\frac{81}{2}$	$-\frac{9}{2}$	-6	$-\frac{9}{2}$
(2222, 14)	$\frac{7}{800}$	$-\frac{133}{2400}$	$-\frac{27}{800}$	$\frac{7}{100}$	$\frac{171}{800}$	$-\frac{28}{75}$	$-\frac{27}{100}$	$\frac{36}{25}$	$-\frac{18}{5}$	$-\frac{24}{5}$	$-\frac{48}{5}$	$-\frac{3}{2}$	$-\frac{81}{2}$	$\frac{9}{2}$	0	$\frac{9}{2}$
(2222, 22)	0	$\frac{7}{75}$	0	$-\frac{7}{100}$	$-\frac{9}{25}$	$-\frac{28}{75}$	$\frac{27}{100}$	$\frac{36}{25}$	0	$-\frac{72}{5}$	$-\frac{288}{5}$	0	0	0	12	0
(2222, 24)	0	$\frac{13}{300}$	0	$-\frac{13}{200}$	$\frac{9}{100}$	$\frac{26}{75}$	$-\frac{27}{200}$	$\frac{18}{25}$	0	$\frac{48}{5}$	$\frac{96}{5}$	0	0	0	-6	0
(2222, 44)	0	$\frac{7}{300}$	0	0	$-\frac{9}{100}$	$-\frac{28}{75}$	0	$\frac{36}{25}$	0	$\frac{12}{5}$	$-\frac{48}{5}$	0	0	0	0	0
(2233, 11)	$\frac{1}{75}$	$-\frac{1}{75}$	$\frac{3}{25}$	$-\frac{1}{150}$	$-\frac{3}{25}$	$\frac{8}{75}$	$-\frac{3}{50}$	$\frac{24}{25}$	$\frac{24}{5}$	48	$\frac{768}{5}$	10	18	0	-24	-6
(2233, 12)	$\frac{1}{120}$	$-\frac{1}{120}$	$-\frac{3}{40}$	$\frac{1}{120}$	$\frac{3}{40}$	$-\frac{2}{15}$	$-\frac{3}{40}$	$\frac{6}{5}$	0	-12	-96	$-\frac{7}{2}$	$-\frac{9}{2}$	3	6	$\frac{9}{2}$
(2233, 14)	$\frac{1}{300}$	$-\frac{1}{300}$	$\frac{3}{100}$	$-\frac{1}{150}$	$-\frac{3}{100}$	$\frac{8}{75}$	$-\frac{3}{50}$	$\frac{24}{25}$	$\frac{21}{5}$	18	$\frac{48}{5}$	$\frac{5}{2}$	$\frac{63}{2}$	0	-12	$-\frac{3}{2}$
(2233, 22)	$\frac{1}{300}$	$-\frac{1}{300}$	$\frac{3}{100}$	$-\frac{1}{150}$	$-\frac{3}{100}$	$\frac{8}{75}$	$-\frac{3}{50}$	$\frac{24}{25}$	$\frac{6}{5}$	12	$\frac{288}{5}$	1	-9	0	0	-3
(2233, 24)	$\frac{1}{480}$	$-\frac{1}{480}$	$-\frac{3}{160}$	$\frac{1}{120}$	$\frac{3}{160}$	$-\frac{2}{15}$	$-\frac{3}{40}$	$\frac{6}{5}$	0	0	0	-2	-18	$\frac{3}{2}$	6	0
(2233, 44)	$\frac{1}{1200}$	$-\frac{1}{1200}$	$\frac{3}{400}$	$-\frac{1}{150}$	$-\frac{3}{400}$	$\frac{8}{75}$	$-\frac{3}{50}$	$\frac{24}{25}$	$\frac{9}{5}$	6	$\frac{48}{5}$	$-\frac{1}{2}$	$\frac{9}{2}$	0	0	$-\frac{3}{2}$
(3333, 11)	$\frac{1}{75}$	$-\frac{1}{100}$	$-\frac{7}{25}$	$-\frac{4}{75}$	$\frac{21}{100}$	0	$\frac{28}{25}$	0	$\frac{24}{5}$	$\frac{96}{5}$	0	0	0	4	0	0
(3333, 12)	$\frac{1}{300}$	$-\frac{1}{200}$	$\frac{13}{100}$	$\frac{2}{75}$	$-\frac{39}{200}$	0	$\frac{26}{25}$	0	$\frac{16}{5}$	$\frac{32}{5}$	0	0	0	2	0	0
(3333, 14)	$\frac{1}{300}$	0	$-\frac{7}{100}$	$-\frac{4}{75}$	0	0	$\frac{28}{25}$	0	$\frac{6}{5}$	$-\frac{24}{5}$	0	0	0	4	0	0
(3333, 22)	$\frac{1}{300}$	0	$-\frac{7}{100}$	$-\frac{4}{75}$	0	0	$\frac{28}{25}$	0	$-\frac{4}{5}$	$\frac{16}{5}$	0	0	0	0	0	0
(3333, 24)	$\frac{1}{1200}$	$-\frac{1}{400}$	$\frac{13}{400}$	$\frac{2}{75}$	$-\frac{39}{400}$	0	$\frac{26}{25}$	0	$\frac{4}{5}$	$\frac{32}{5}$	0	0	0	-1	0	0
(3333, 44)	$\frac{1}{1200}$	$\frac{1}{400}$	$-\frac{7}{400}$	$-\frac{4}{75}$	$-\frac{21}{400}$	0	$\frac{28}{25}$	0	$-\frac{6}{5}$	$-\frac{24}{5}$	0	0	0	1	0	0

Table 4. (Theorem 2.1 (ii))

$a_1 a_2 a_3$ $a_4, b_1 b_2$	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}	β_{11}	β_{12}	β_{13}	β_{14}
1112, 11	$\frac{-26}{451}$	$\frac{108}{451}$	$\frac{6656}{451}$	$\frac{27648}{451}$	$\frac{168}{451}$	$\frac{11448}{451}$	$\frac{-2496}{451}$	$\frac{-17280}{451}$	$\frac{24}{41}$	$\frac{936}{41}$	$\frac{144}{41}$	$\frac{-384}{41}$	$\frac{4032}{41}$	$\frac{-48}{41}$
1112, 12	$\frac{28}{451}$	$\frac{54}{451}$	$\frac{3584}{451}$	$\frac{-6912}{451}$	$\frac{480}{451}$	$\frac{-2052}{451}$	$\frac{-2688}{451}$	$\frac{1728}{451}$	$\frac{-60}{41}$	$\frac{216}{41}$	$\frac{-108}{41}$	$\frac{-2112}{41}$	$\frac{-1440}{41}$	$\frac{288}{41}$
1112, 14	$\frac{-26}{451}$	$\frac{108}{451}$	$\frac{1664}{451}$	$\frac{6912}{451}$	$\frac{-912}{451}$	$\frac{3672}{451}$	0	$\frac{-6912}{451}$	$\frac{-48}{41}$	$\frac{-54}{41}$	$\frac{702}{41}$	$\frac{1632}{41}$	$\frac{576}{41}$	$\frac{-228}{41}$
1112, 22	$\frac{-26}{451}$	$\frac{108}{451}$	$\frac{1664}{451}$	$\frac{6912}{451}$	$\frac{-912}{451}$	$\frac{3672}{451}$	0	$\frac{-6912}{451}$	$\frac{-48}{41}$	$\frac{684}{41}$	$\frac{-36}{41}$	$\frac{-2304}{41}$	$\frac{576}{41}$	$\frac{264}{41}$
1112, 24	$\frac{28}{451}$	$\frac{54}{451}$	$\frac{896}{451}$	$\frac{-1728}{451}$	$\frac{-66}{41}$	$\frac{-108}{451}$	$\frac{-1344}{451}$	$\frac{-864}{451}$	$\frac{-42}{41}$	$\frac{-90}{41}$	$\frac{306}{41}$	$\frac{336}{41}$	$\frac{-576}{41}$	$\frac{-36}{41}$
1112, 44	$\frac{-26}{451}$	$\frac{108}{451}$	$\frac{416}{451}$	$\frac{1728}{451}$	$\frac{-1182}{451}$	$\frac{1728}{451}$	$\frac{624}{451}$	$\frac{-4320}{451}$	$\frac{-66}{41}$	$\frac{252}{41}$	$\frac{288}{41}$	$\frac{-816}{41}$	$\frac{-288}{41}$	$\frac{96}{41}$
1222, 11	$\frac{-26}{451}$	$\frac{108}{451}$	$\frac{3328}{451}$	$\frac{13824}{451}$	$\frac{8468}{451}$	$\frac{6264}{451}$	$\frac{-15264}{451}$	$\frac{-10368}{451}$	$\frac{-352}{41}$	$\frac{-708}{41}$	$\frac{516}{41}$	$\frac{3584}{41}$	$\frac{13536}{41}$	$\frac{-660}{41}$
1222, 12	$\frac{28}{451}$	$\frac{54}{451}$	$\frac{1792}{451}$	$\frac{-3456}{451}$	$\frac{9598}{451}$	$\frac{-756}{451}$	$\frac{-16224}{451}$	0	$\frac{-458}{41}$	$\frac{-1956}{41}$	$\frac{168}{41}$	$\frac{2800}{41}$	$\frac{5040}{41}$	$\frac{-420}{41}$
1222, 14	$\frac{-26}{451}$	$\frac{108}{451}$	$\frac{832}{451}$	$\frac{3456}{451}$	$\frac{-6955}{451}$	$\frac{216}{41}$	$\frac{7632}{451}$	$\frac{-5184}{451}$	$\frac{473}{41}$	$\frac{1749}{41}$	$\frac{57}{41}$	-56	-144	$\frac{357}{41}$
1222, 22	$\frac{-26}{451}$	$\frac{108}{451}$	$\frac{832}{451}$	$\frac{3456}{451}$	$\frac{10634}{451}$	$\frac{216}{41}$	$\frac{-14016}{451}$	$\frac{-5184}{451}$	$\frac{-634}{41}$	$\frac{-2310}{41}$	$\frac{426}{41}$	112	288	$\frac{-750}{41}$
1222, 24	$\frac{28}{451}$	$\frac{54}{451}$	$\frac{448}{451}$	$\frac{-864}{451}$	$\frac{-3182}{451}$	$\frac{216}{451}$	$\frac{6096}{451}$	$\frac{-1296}{451}$	$\frac{166}{41}$	$\frac{474}{41}$	$\frac{6}{41}$	$\frac{-896}{41}$	$\frac{-3384}{41}$	$\frac{156}{41}$
1222, 44	$\frac{-26}{451}$	$\frac{108}{451}$	$\frac{208}{451}$	$\frac{864}{451}$	$\frac{2381}{451}$	$\frac{1404}{451}$	$\frac{-2880}{451}$	$\frac{-3888}{451}$	$\frac{-151}{41}$	$\frac{-681}{41}$	$\frac{219}{41}$	$\frac{1400}{41}$	$\frac{2520}{41}$	$\frac{-219}{41}$
1233, 11	$\frac{10}{451}$	$\frac{72}{451}$	$\frac{2560}{451}$	$\frac{-18432}{451}$	$\frac{488}{451}$	$\frac{-6192}{451}$	$\frac{-1600}{451}$	$\frac{6912}{451}$	$\frac{-112}{41}$	$\frac{-480}{41}$	$\frac{432}{41}$	$\frac{704}{41}$	$\frac{-3456}{41}$	$\frac{-24}{41}$
1233, 12	$\frac{-8}{451}$	$\frac{90}{451}$	$\frac{1024}{451}$	$\frac{11520}{451}$	$\frac{-1280}{451}$	$\frac{5220}{451}$	$\frac{-256}{451}$	$\frac{-8640}{451}$	$\frac{-20}{41}$	$\frac{312}{41}$	$\frac{348}{41}$	$\frac{-512}{41}$	$\frac{1440}{41}$	$\frac{24}{41}$
1233, 14	$\frac{10}{451}$	$\frac{72}{451}$	$\frac{640}{451}$	$\frac{-4608}{451}$	$\frac{-760}{451}$	$\frac{-1008}{451}$	$\frac{-640}{451}$	0	$\frac{-64}{41}$	$\frac{-66}{41}$	$\frac{306}{41}$	$\frac{-640}{41}$	$\frac{-1152}{41}$	$\frac{96}{41}$
1233, 22	$\frac{10}{451}$	$\frac{72}{451}$	$\frac{640}{451}$	$\frac{-4608}{451}$	$\frac{-760}{451}$	$\frac{-1008}{451}$	$\frac{-640}{451}$	0	$\frac{-64}{41}$	$\frac{180}{41}$	$\frac{60}{41}$	$\frac{-640}{41}$	$\frac{-1152}{41}$	$\frac{96}{41}$
1233, 24	$\frac{-8}{451}$	$\frac{90}{451}$	$\frac{256}{451}$	$\frac{2880}{451}$	$\frac{-1238}{451}$	$\frac{180}{41}$	$\frac{128}{451}$	$\frac{-4320}{451}$	$\frac{-50}{41}$	$\frac{330}{41}$	$\frac{150}{41}$	-16	0	$\frac{72}{41}$
1233, 44	$\frac{10}{451}$	$\frac{72}{451}$	$\frac{160}{451}$	$\frac{-1152}{451}$	$\frac{-1072}{451}$	$\frac{288}{451}$	$\frac{-400}{451}$	$\frac{-1728}{451}$	$\frac{-52}{41}$	$\frac{222}{41}$	$\frac{90}{41}$	$\frac{-976}{41}$	$\frac{-576}{41}$	$\frac{126}{41}$

Table 5. (Theorem 2.1 (iii))

$(a_1 a_2 a_3 a_4, b_1 b_2)$	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_7	γ_8	γ_9	γ_{10}	γ_{11}	γ_{12}	γ_{13}	γ_{14}	γ_{15}	γ_{16}
(1113, 11)	$\frac{1}{23}$	$\frac{288}{23}$	$\frac{32}{23}$	$\frac{9}{23}$	0	0	0	0	$\frac{84}{23}$	$\frac{720}{23}$	$\frac{336}{23}$	$\frac{864}{23}$	0	0	0	0
(1113, 12)	$\frac{1}{23}$	$\frac{144}{23}$	$\frac{-16}{23}$	$\frac{-9}{23}$	0	0	0	0	$\frac{156}{23}$	$\frac{-48}{23}$	$\frac{-168}{23}$	$\frac{-456}{23}$	0	0	0	0
(1113, 14)	$\frac{1}{23}$	$\frac{72}{23}$	$\frac{8}{23}$	$\frac{9}{23}$	0	0	0	0	$\frac{186}{23}$	$\frac{600}{23}$	$\frac{-228}{23}$	$\frac{372}{23}$	0	0	0	0
(1113, 22)	$\frac{1}{23}$	$\frac{72}{23}$	$\frac{8}{23}$	$\frac{9}{23}$	0	0	0	0	$\frac{48}{23}$	$\frac{48}{23}$	$\frac{48}{23}$	$\frac{96}{23}$	0	0	0	0
(1113, 24)	$\frac{1}{23}$	$\frac{36}{23}$	$\frac{-4}{23}$	$\frac{-9}{23}$	0	0	0	0	$\frac{114}{23}$	$\frac{84}{23}$	$\frac{-120}{23}$	$\frac{-156}{23}$	0	0	0	0
(1113, 44)	$\frac{1}{23}$	$\frac{18}{23}$	$\frac{2}{23}$	$\frac{9}{23}$	0	0	0	0	$\frac{108}{23}$	$\frac{156}{23}$	$\frac{-162}{23}$	$\frac{42}{23}$	0	0	0	0
(1223, 11)	0	$\frac{144}{23}$	$\frac{16}{23}$	0	$\frac{1}{23}$	0	0	$\frac{-9}{23}$	$\frac{162}{23}$	$\frac{264}{23}$	$\frac{-1188}{23}$	$\frac{420}{23}$	$\frac{192}{23}$	$\frac{864}{23}$	$\frac{-4704}{23}$	$\frac{-5760}{23}$
(1223, 12)	0	$\frac{72}{23}$	$\frac{-8}{23}$	0	$\frac{1}{23}$	0	0	$\frac{9}{23}$	$\frac{120}{23}$	$\frac{96}{23}$	$\frac{336}{23}$	$\frac{-384}{23}$	$\frac{-240}{23}$	$\frac{-912}{23}$	$\frac{4368}{23}$	$\frac{2784}{23}$
(1223, 14)	0	$\frac{36}{23}$	$\frac{4}{23}$	0	$\frac{1}{23}$	0	0	$\frac{-9}{23}$	$\frac{144}{23}$	$\frac{480}{23}$	$\frac{-504}{23}$	$\frac{312}{23}$	$\frac{-360}{23}$	$\frac{1416}{23}$	$\frac{-1944}{23}$	$\frac{-1344}{23}$
(1223, 22)	0	$\frac{36}{23}$	$\frac{4}{23}$	0	$\frac{1}{23}$	0	0	$\frac{-9}{23}$	$\frac{6}{23}$	$\frac{-72}{23}$	$\frac{-228}{23}$	$\frac{36}{23}$	$\frac{192}{23}$	$\frac{-240}{23}$	$\frac{-1392}{23}$	$\frac{-1344}{23}$
(1223, 24)	0	$\frac{18}{23}$	$\frac{-2}{23}$	0	$\frac{1}{23}$	0	0	$\frac{9}{23}$	$\frac{30}{23}$	$\frac{24}{23}$	$\frac{84}{23}$	$\frac{-96}{23}$	$\frac{36}{23}$	$\frac{-360}{23}$	$\frac{504}{23}$	$\frac{576}{23}$
(1223, 44)	0	$\frac{9}{23}$	$\frac{1}{23}$	0	$\frac{1}{23}$	0	0	$\frac{-9}{23}$	$\frac{36}{23}$	$\frac{120}{23}$	$\frac{-126}{23}$	$\frac{78}{23}$	$\frac{-84}{23}$	$\frac{312}{23}$	$\frac{-840}{23}$	$\frac{-240}{23}$
(1333, 11)	$\frac{1}{23}$	$\frac{96}{23}$	$\frac{-32}{23}$	$\frac{-3}{23}$	0	0	0	0	$\frac{260}{23}$	$\frac{352}{23}$	$\frac{-544}{23}$	$\frac{-352}{23}$	0	0	0	0
(1333, 12)	$\frac{1}{23}$	$\frac{48}{23}$	$\frac{16}{23}$	$\frac{3}{23}$	0	0	0	0	$\frac{116}{23}$	$\frac{160}{23}$	$\frac{-40}{23}$	$\frac{104}{23}$	0	0	0	0
(1333, 14)	$\frac{1}{23}$	$\frac{24}{23}$	$\frac{-8}{23}$	$\frac{-3}{23}$	0	0	0	0	$\frac{170}{23}$	$\frac{16}{23}$	$\frac{-292}{23}$	$\frac{-340}{23}$	0	0	0	0
(1333, 22)	$\frac{1}{23}$	$\frac{24}{23}$	$\frac{-8}{23}$	$\frac{-3}{23}$	0	0	0	0	$\frac{32}{23}$	$\frac{16}{23}$	$\frac{-16}{23}$	$\frac{-64}{23}$	0	0	0	0
(1333, 24)	$\frac{1}{23}$	$\frac{12}{23}$	$\frac{4}{23}$	$\frac{3}{23}$	0	0	0	0	$\frac{26}{23}$	$\frac{76}{23}$	$\frac{32}{23}$	$\frac{116}{23}$	0	0	0	0
(1333, 44)	$\frac{1}{23}$	$\frac{6}{23}$	$\frac{-2}{23}$	$\frac{-3}{23}$	0	0	0	0	$\frac{44}{23}$	$\frac{-68}{23}$	$\frac{-22}{23}$	$\frac{-130}{23}$	0	0	0	0

Table 6. (Theorem 2.1 (iv))

$a_1 a_2 a_3$ $a_4 b_1 b_2$	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	δ_7	δ_8	δ_9	δ_{10}	δ_{11}	δ_{12}	δ_{13}	δ_{14}
112311	$\frac{1}{261}$	$\frac{256}{29}$	$\frac{-256}{261}$	$\frac{-1}{29}$	$\frac{1808}{87}$	$\frac{656}{29}$	$\frac{-2056}{87}$	$\frac{-3808}{29}$	$\frac{-4144}{29}$	$\frac{736}{3}$	$\frac{472}{3}$	$\frac{-41984}{87}$	$\frac{-1096}{87}$	$\frac{-968}{87}$
112312	$\frac{1}{261}$	$\frac{128}{29}$	$\frac{128}{261}$	$\frac{1}{29}$	$\frac{208}{87}$	$\frac{-32}{29}$	$\frac{-284}{87}$	$\frac{-368}{29}$	$\frac{1048}{29}$	$\frac{-6224}{87}$	$\frac{-7100}{87}$	$\frac{21248}{87}$	$\frac{8}{3}$	$\frac{500}{87}$
112314	$\frac{1}{261}$	$\frac{64}{29}$	$\frac{-64}{261}$	$\frac{-1}{29}$	$\frac{-84}{29}$	$\frac{-114}{29}$	$\frac{450}{29}$	$\frac{1800}{29}$	$\frac{-324}{29}$	$\frac{-4200}{29}$	$\frac{-2814}{29}$	$\frac{-3072}{29}$	$\frac{318}{29}$	$\frac{414}{29}$
112322	$\frac{1}{261}$	$\frac{64}{29}$	$\frac{-64}{261}$	$\frac{-1}{29}$	$\frac{264}{29}$	$\frac{60}{29}$	$\frac{-420}{29}$	$\frac{-1680}{29}$	$\frac{-1368}{29}$	$\frac{2064}{29}$	$\frac{1884}{29}$	$\frac{-3072}{29}$	$\frac{-204}{29}$	$\frac{-108}{29}$
112324	$\frac{1}{261}$	$\frac{32}{29}$	$\frac{32}{261}$	$\frac{1}{29}$	$\frac{860}{87}$	$\frac{218}{29}$	$\frac{-970}{87}$	$\frac{-2296}{29}$	$\frac{-628}{29}$	$\frac{7976}{87}$	$\frac{-778}{87}$	$\frac{4288}{87}$	$\frac{-622}{87}$	$\frac{-10}{3}$
112344	$\frac{1}{261}$	$\frac{16}{29}$	$\frac{-16}{261}$	$\frac{-1}{29}$	$\frac{16}{87}$	$\frac{-176}{29}$	$\frac{244}{87}$	$\frac{592}{29}$	$\frac{-152}{29}$	$\frac{-6992}{87}$	$\frac{-3404}{87}$	$\frac{-1024}{87}$	$\frac{292}{87}$	$\frac{620}{87}$
222311	$\frac{1}{261}$	$\frac{128}{29}$	$\frac{-128}{261}$	$\frac{-1}{29}$	$\frac{5480}{261}$	$\frac{1472}{87}$	$\frac{-15016}{261}$	$\frac{-2992}{87}$	$\frac{-23584}{87}$	$\frac{79664}{261}$	$\frac{102248}{261}$	$\frac{-194048}{261}$	$\frac{-116}{9}$	$\frac{-3704}{261}$
222312	$\frac{1}{261}$	$\frac{64}{29}$	$\frac{64}{261}$	$\frac{1}{29}$	$\frac{-160}{261}$	$\frac{-640}{87}$	$\frac{7172}{261}$	$\frac{-5656}{87}$	$\frac{12320}{87}$	$\frac{-50824}{261}$	$\frac{-47284}{261}$	$\frac{96640}{261}$	$\frac{1076}{261}$	$\frac{964}{261}$
222314	$\frac{1}{261}$	$\frac{32}{29}$	$\frac{-32}{261}$	$\frac{-1}{29}$	$\frac{824}{261}$	$\frac{-466}{87}$	$\frac{4970}{261}$	$\frac{-1888}{87}$	$\frac{-628}{87}$	$\frac{-24496}{261}$	$\frac{-10030}{261}$	$\frac{-44672}{261}$	$\frac{494}{261}$	$\frac{394}{261}$
222322	$\frac{1}{261}$	$\frac{32}{29}$	$\frac{-32}{261}$	$\frac{-1}{29}$	$\frac{824}{261}$	$\frac{1100}{87}$	$\frac{-9124}{261}$	$\frac{2288}{87}$	$\frac{-10024}{87}$	$\frac{50672}{261}$	$\frac{32252}{261}$	$\frac{-44672}{261}$	$\frac{-1072}{261}$	$\frac{-1172}{261}$
222324	$\frac{1}{261}$	$\frac{16}{29}$	$\frac{16}{261}$	$\frac{1}{29}$	$\frac{-748}{261}$	$\frac{518}{87}$	$\frac{-2470}{261}$	$\frac{2936}{87}$	$\frac{-1156}{87}$	$\frac{11192}{261}$	$\frac{-8830}{261}$	$\frac{21088}{261}$	$\frac{578}{261}$	$\frac{562}{261}$
222344	$\frac{1}{261}$	$\frac{8}{29}$	$\frac{-8}{261}$	$\frac{-1}{29}$	$\frac{-340}{261}$	$\frac{224}{87}$	$\frac{-604}{261}$	$\frac{1520}{87}$	$\frac{-1936}{87}$	$\frac{5840}{261}$	$\frac{-6388}{261}$	$\frac{-7328}{261}$	$\frac{284}{261}$	$\frac{244}{261}$
233311	$\frac{1}{261}$	$\frac{256}{87}$	$\frac{256}{261}$	$\frac{1}{87}$	$\frac{-13360}{261}$	$\frac{-3520}{87}$	$\frac{20816}{261}$	$\frac{12608}{29}$	$\frac{20960}{87}$	$\frac{-219712}{261}$	$\frac{-113968}{261}$	$\frac{133120}{261}$	$\frac{15464}{261}$	$\frac{16384}{261}$
233312	$\frac{1}{261}$	$\frac{128}{87}$	$\frac{-128}{261}$	$\frac{-1}{87}$	$\frac{11168}{261}$	$\frac{2312}{87}$	$\frac{-14212}{261}$	$\frac{-27920}{87}$	$\frac{-14968}{87}$	$\frac{129296}{261}$	$\frac{45212}{261}$	$\frac{-64256}{261}$	$\frac{-9856}{261}$	$\frac{-8948}{261}$
233314	$\frac{1}{261}$	$\frac{64}{87}$	$\frac{64}{261}$	$\frac{1}{87}$	$\frac{3836}{261}$	$\frac{338}{87}$	$\frac{-4126}{261}$	$\frac{-2968}{29}$	$\frac{-2092}{87}$	$\frac{21368}{261}$	$\frac{-10174}{261}$	$\frac{32512}{261}$	$\frac{-2530}{261}$	$\frac{-1514}{261}$
233322	$\frac{1}{261}$	$\frac{64}{87}$	$\frac{64}{261}$	$\frac{1}{87}$	$\frac{-5560}{261}$	$\frac{-1228}{87}$	$\frac{6836}{261}$	$\frac{4688}{29}$	$\frac{7304}{87}$	$\frac{-72592}{261}$	$\frac{-24268}{261}$	$\frac{32512}{261}$	$\frac{5300}{261}$	$\frac{6316}{261}$
233324	$\frac{1}{261}$	$\frac{32}{87}$	$\frac{-32}{261}$	$\frac{-1}{87}$	$\frac{-940}{261}$	$\frac{-274}{87}$	$\frac{842}{261}$	$\frac{2584}{87}$	$\frac{-76}{87}$	$\frac{-12808}{261}$	$\frac{2666}{261}$	$\frac{-14528}{261}$	$\frac{878}{261}$	$\frac{1882}{261}$
233344	$\frac{1}{261}$	$\frac{16}{87}$	$\frac{16}{261}$	$\frac{1}{87}$	$\frac{1088}{261}$	$\frac{128}{87}$	$\frac{-2140}{261}$	$\frac{-1120}{29}$	$\frac{-808}{87}$	$\frac{11168}{261}$	$\frac{5204}{261}$	$\frac{7360}{261}$	$\frac{-1156}{261}$	$\frac{-4}{9}$

Table 7. (Theorem 2.2)

$(1, c_1, c_2, c_3)$	ν_1	ν_2	ν_3	ν_4	ν_5	ν_6	ν_7	ν_8	ν_9	ν_{10}	ν_{11}	ν_{12}	ν_{13}	ν_{14}	ν_{15}	ν_{16}
$(1, 1, 1, 2)$	$\frac{3}{40}$	$-\frac{1}{5}$	$-\frac{27}{40}$	0	$\frac{9}{5}$	0	0	0	0	0	0	0	0	0	0	0
$(1, 1, 1, 4)$	$\frac{3}{100}$	$-\frac{9}{100}$	$\frac{27}{100}$	$\frac{4}{25}$	$-\frac{81}{100}$	0	$\frac{36}{25}$	0	$\frac{54}{5}$	$\frac{432}{5}$	0	0	0	0	0	0
$(1, 1, 1, 8)$	$\frac{3}{160}$	$-\frac{9}{160}$	$-\frac{27}{160}$	$\frac{9}{80}$	$\frac{81}{160}$	$-\frac{1}{5}$	$-\frac{81}{80}$	$\frac{9}{5}$	0	0	0	$-\frac{27}{4}$	$\frac{243}{4}$	0	81	$\frac{81}{4}$
$(1, 1, 2, 2)$	$\frac{1}{50}$	$\frac{2}{25}$	$\frac{9}{50}$	0	$\frac{18}{25}$	0	0	0	$\frac{36}{5}$	0	0	0	0	0	0	0
$(1, 1, 2, 4)$	$\frac{1}{80}$	$\frac{1}{16}$	$-\frac{9}{80}$	$-\frac{1}{5}$	$-\frac{9}{16}$	0	$\frac{9}{5}$	0	0	0	0	0	0	9	0	0
$(1, 1, 2, 8)$	$\frac{1}{200}$	$\frac{1}{40}$	$\frac{9}{200}$	$-\frac{9}{100}$	$\frac{9}{40}$	$\frac{4}{25}$	$-\frac{81}{100}$	$\frac{36}{25}$	$\frac{9}{5}$	$\frac{144}{5}$	$\frac{1152}{5}$	9	81	0	0	0
$(1, 1, 4, 4)$	$\frac{1}{200}$	$\frac{3}{200}$	$\frac{9}{200}$	$\frac{2}{25}$	$\frac{27}{200}$	0	$\frac{18}{25}$	0	$\frac{54}{5}$	$\frac{216}{5}$	0	0	0	0	0	0
$(1, 1, 4, 8)$	$\frac{1}{320}$	$\frac{3}{320}$	$\frac{9}{320}$	$\frac{1}{16}$	$-\frac{27}{320}$	$-\frac{1}{5}$	$-\frac{9}{16}$	$\frac{9}{5}$	0	0	0	$\frac{9}{4}$	$-\frac{81}{4}$	$\frac{9}{4}$	27	$\frac{27}{4}$
$(1, 1, 8, 8)$	$\frac{1}{800}$	$\frac{3}{800}$	$\frac{9}{800}$	$\frac{3}{200}$	$\frac{27}{800}$	$\frac{2}{25}$	$\frac{27}{200}$	$\frac{18}{25}$	$\frac{36}{5}$	$\frac{234}{5}$	$\frac{576}{5}$	$\frac{9}{2}$	$\frac{81}{2}$	0	0	0
$(1, 2, 2, 2)$	$\frac{1}{40}$	$-\frac{3}{20}$	$-\frac{9}{40}$	0	$\frac{27}{20}$	0	0	0	0	0	0	0	0	0	0	0
$(1, 2, 2, 4)$	$\frac{1}{100}$	$-\frac{7}{100}$	$\frac{9}{100}$	$\frac{4}{25}$	$-\frac{63}{100}$	0	$\frac{36}{25}$	0	$\frac{18}{5}$	$\frac{72}{5}$	0	0	0	0	0	0
$(1, 2, 2, 8)$	$\frac{1}{160}$	$-\frac{7}{160}$	$-\frac{9}{160}$	$\frac{9}{80}$	$\frac{63}{160}$	$-\frac{1}{5}$	$-\frac{81}{80}$	$\frac{9}{5}$	0	0	0	$-\frac{9}{4}$	$\frac{81}{4}$	0	9	$\frac{27}{4}$
$(1, 2, 4, 4)$	$\frac{1}{160}$	$-\frac{1}{32}$	$-\frac{9}{160}$	$-\frac{1}{10}$	$\frac{9}{32}$	0	$\frac{9}{10}$	0	0	0	0	0	0	$\frac{9}{2}$	0	0
$(1, 2, 4, 8)$	$\frac{1}{400}$	$-\frac{1}{80}$	$\frac{9}{400}$	$-\frac{1}{20}$	$-\frac{9}{80}$	$\frac{4}{25}$	$-\frac{9}{20}$	$\frac{36}{25}$	$\frac{9}{10}$	$\frac{27}{5}$	$\frac{72}{5}$	$\frac{9}{2}$	$\frac{81}{2}$	0	0	0
$(1, 2, 8, 8)$	$\frac{1}{640}$	$-\frac{1}{128}$	$-\frac{9}{640}$	$-\frac{3}{160}$	$\frac{9}{128}$	$-\frac{1}{10}$	$\frac{27}{160}$	$\frac{9}{10}$	0	0	0	$-\frac{9}{8}$	$\frac{81}{8}$	$\frac{27}{8}$	$\frac{9}{2}$	$\frac{27}{8}$
$(1, 4, 4, 4)$	$\frac{1}{400}$	$-\frac{9}{400}$	$\frac{9}{400}$	$\frac{3}{25}$	$-\frac{81}{400}$	0	$\frac{27}{25}$	0	$\frac{27}{5}$	$\frac{54}{5}$	0	0	0	0	0	0
$(1, 4, 4, 8)$	$\frac{1}{640}$	$-\frac{9}{640}$	$-\frac{9}{640}$	$\frac{7}{80}$	$\frac{81}{640}$	$-\frac{1}{5}$	$-\frac{63}{80}$	$\frac{9}{5}$	0	0	0	$\frac{9}{8}$	$-\frac{81}{8}$	$\frac{9}{8}$	0	$\frac{27}{8}$
$(1, 4, 8, 8)$	$\frac{1}{1600}$	$-\frac{9}{1600}$	$\frac{9}{1600}$	$\frac{1}{40}$	$-\frac{81}{1600}$	$\frac{2}{25}$	$\frac{9}{40}$	$\frac{18}{25}$	$\frac{18}{5}$	$\frac{36}{5}$	$\frac{36}{5}$	$\frac{9}{4}$	$\frac{81}{4}$	0	0	0
$(1, 8, 8, 8)$	$\frac{1}{2560}$	$-\frac{9}{2560}$	$-\frac{9}{2560}$	$\frac{9}{320}$	$\frac{81}{2560}$	$-\frac{3}{20}$	$-\frac{81}{320}$	$\frac{27}{20}$	0	0	0	$\frac{27}{32}$	$-\frac{243}{32}$	$\frac{81}{32}$	0	$\frac{81}{32}$

5.2. Sample formulas. In this section we shall give explicit formulas for a few cases from Tables 1 and 2.

First two formulas of Theorem 2.1(i):

$$\begin{aligned} N(1, 1, 1, 1, 1, 1; n) &= \frac{112}{5}\sigma_3(n) - \frac{84}{5}\sigma_3(n/2) - \frac{432}{5}\sigma_3(n/3) - \frac{448}{5}\sigma_3(n/4) + \frac{324}{5}\sigma_3(n/6) \\ &\quad + \frac{1728}{5}\sigma_3(n/12) - \frac{72}{5}a_{4,6}(n) - \frac{288}{5}a_{4,6}(n/2) + 12a_{4,12}(n), \end{aligned}$$

$$\begin{aligned} N(1, 1, 1, 1, 1, 2; n) &= \frac{52}{5}\sigma_3(n) - \frac{78}{5}\sigma_3(n/2) + \frac{108}{5}\sigma_3(n/3) + \frac{416}{5}\sigma_3(n/4) - \frac{324}{5}\sigma_3(n/6) \\ &\quad + \frac{864}{5}\sigma_3(n/12) + \frac{48}{5}a_{4,6}(n) + \frac{96}{5}a_{4,6}(n/2) - 6a_{4,12}(n). \end{aligned}$$

First two formulas of Theorem 2.1(ii):

$$\begin{aligned} N(1, 1, 1, 2, 1, 1; n) &= -\frac{26}{451}\sigma_{3;1,\chi_8}(n) + \frac{108}{451}\sigma_{3;1,\chi_8}(n/3) + \frac{6656}{451}\sigma_{3;\chi_8,1}(n) + \frac{27648}{451}\sigma_{3;\chi_8,1}(n/3) \\ &\quad + \frac{168}{451}a_{4,8,\chi_8;1}(n) + \frac{11448}{451}a_{4,8,\chi_8;1}(n/3) - \frac{2496}{451}a_{4,8,\chi_8;2}(n) - \frac{17280}{451}a_{4,8,\chi_8;2}(n/3) \\ &\quad + \frac{24}{41}a_{4,24,\chi_8;1}(n) + \frac{936}{41}a_{4,24,\chi_8;2}(n) + \frac{144}{41}a_{4,24,\chi_8;3}(n) - \frac{384}{41}a_{4,24,\chi_8;4}(n) \\ &\quad + \frac{4032}{41}a_{4,24,\chi_8;5}(n) - \frac{48}{41}a_{4,24,\chi_8;6}(n), \end{aligned}$$

$$\begin{aligned}
N(1, 1, 1, 2, 1, 2; n) &= \frac{28}{451}\sigma_{3;1,\chi_8}(n) + \frac{54}{451}\sigma_{3;1,\chi_8}(n/3) + \frac{3584}{451}\sigma_{3;\chi_8,1}(n) - \frac{6912}{451}\sigma_{3;\chi_8,1}(n/3) \\
&+ \frac{480}{451}a_{4,8,\chi_8;1}(n) - \frac{2052}{451}a_{4,8,\chi_8;1}(n/3) - \frac{2688}{451}a_{4,8,\chi_8;2}(n) + \frac{1728}{451}a_{4,8,\chi_8;2}(n/3) \\
&- \frac{60}{41}a_{4,24,\chi_8;1}(n) + \frac{216}{41}a_{4,24,\chi_8;2}(n) - \frac{108}{41}a_{4,24,\chi_8;3}(n) - \frac{2112}{41}a_{4,24,\chi_8;4}(n) \\
&- \frac{1440}{41}a_{4,24,\chi_8;5}(n) + \frac{288}{41}a_{4,24,\chi_8;6}(n).
\end{aligned}$$

First two formulas of Theorem 2.1(iii):

$$\begin{aligned}
N(1, 1, 1, 3, 1, 1; n) &= \frac{1}{23}\sigma_{3;1,\chi_{12}}(n) + \frac{288}{23}\sigma_{3;\chi_{12},1}(n) + \frac{32}{23}\sigma_{3;\chi_{-4},\chi_{-3}}(n) + \frac{9}{23}\sigma_{3;\chi_{-3},\chi_{-4}}(n) \\
&+ \frac{84}{23}a_{4,12,\chi_{12};1}(n) + \frac{720}{23}a_{4,12,\chi_{12};2}(n) + \frac{336}{23}a_{4,12,\chi_{12};3}(n) + \frac{864}{23}a_{4,12,\chi_{12};4}(n), \\
N(1, 1, 1, 3, 1, 2; n) &= \frac{1}{23}\sigma_{3;1,\chi_{12}}(n) + \frac{144}{23}\sigma_{3;\chi_{12},1}(n) - \frac{16}{23}\sigma_{3;\chi_{-4},\chi_{-3}}(n) - \frac{9}{23}\sigma_{3;\chi_{-3},\chi_{-4}}(n) \\
&+ \frac{156}{23}a_{4,12,\chi_{12};1}(n) - \frac{48}{23}a_{4,12,\chi_{12};2}(n) - \frac{168}{23}a_{4,12,\chi_{12};3}(n) - \frac{456}{23}a_{4,12,\chi_{12};4}(n).
\end{aligned}$$

First two formulas of Theorem 2.1(iv):

$$\begin{aligned}
N(1, 1, 2, 3, 1, 1; n) &= \frac{1}{261}\sigma_{3;1,\chi_{24}}(n) + \frac{256}{29}\sigma_{3;\chi_{24},1}(n) - \frac{256}{261}\sigma_{3;\chi_{-8},\chi_{-3}}(n) - \frac{1}{29}\sigma_{3;\chi_{-3},\chi_{-8}}(n) \\
&+ \frac{1808}{87}a_{4,24,\chi_{24};1}(n) + \frac{656}{29}a_{4,24,\chi_{24};2}(n) - \frac{2056}{87}a_{4,24,\chi_{24};3}(n) - \frac{3808}{29}a_{4,24,\chi_{24};4}(n) \\
&- \frac{4144}{29}a_{4,24,\chi_{24};5}(n) + \frac{736}{3}a_{4,24,\chi_{24};6}(n) + \frac{472}{3}a_{4,24,\chi_{24};7}(n) - \frac{41984}{87}a_{4,24,\chi_{24};8}(n) \\
&- \frac{1096}{87}a_{4,24,\chi_{24};9}(n) - \frac{968}{87}a_{4,24,\chi_{24};10}(n), \\
N(1, 1, 2, 3, 1, 2; n) &= \frac{1}{261}\sigma_{3;1,\chi_{24}}(n) + \frac{128}{29}\sigma_{3;\chi_{24},1}(n) + \frac{128}{261}\sigma_{3;\chi_{-8},\chi_{-3}}(n) + \frac{1}{29}\sigma_{3;\chi_{-3},\chi_{-8}}(n) \\
&+ \frac{208}{87}a_{4,24,\chi_{24};1}(n) - \frac{32}{29}a_{4,24,\chi_{24};2}(n) - \frac{284}{87}a_{4,24,\chi_{24};3}(n) - \frac{368}{29}a_{4,24,\chi_{24};4}(n) \\
&+ \frac{1048}{29}a_{4,24,\chi_{24};5}(n) - \frac{6224}{87}a_{4,24,\chi_{24};6}(n) - \frac{7100}{87}a_{4,24,\chi_{24};7}(n) + \frac{21248}{87}a_{4,24,\chi_{24};8}(n) \\
&+ \frac{8}{3}a_{4,24,\chi_{24};9}(n) + \frac{500}{87}a_{4,24,\chi_{24};10}(n).
\end{aligned}$$

First two formulas of Theorem 2.2:

$$\begin{aligned}
M(1, 1, 1, 2; n) &= 18\sigma_3(n) - 48\sigma_3(n/2) - 162\sigma_3(n/3) + 432\sigma_3(n/6), \\
M(1, 1, 1, 4; n) &= \frac{36}{5}\sigma_3(n) - 48\sigma_3(n/2) + \frac{324}{5}\sigma_3(n/3) + \frac{192}{5}\sigma_3(n/4) - \frac{972}{5}\sigma_3(n/6) \\
&+ \frac{1728}{5}\sigma_3(n/12) + \frac{54}{5}a_{4,6}(n) + \frac{432}{5}a_{4,6}(n/2).
\end{aligned}$$

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